Midterm- Computer Science 2 (2021-22)
Time: 2 hours.
Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. How is the number -100.625 stored as a floating point number in the computer ? Give the sign, mantissa and exponent. **[4 marks]**

2. You have to give proper explanations for the questions below, and not just write down the values.

   (a) Find the smallest possible floating point number. **[2 marks]**

   (b) Find the value of Machine ϵ, that is the smallest number ϵ such that $1+\epsilon > 1$ in the floating point number system. **[2 marks]**

3. Consider the solution to $x = \log(3x+1)$ in $[1,3]$. Consider the iterations $x_{k+1} = \log(3x_k+1)$. Starting from $x_0 = \frac{3}{2}$ how many iterations are necessary before we are within 10^{-6} of the solution? **[4 marks]**

4. Consider the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ given below.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}$$

   (a) Find the QR decomposition of $\mathbf{A}$. **[4 marks]**

   (b) Find the least squares solution of the system $\mathbf{Ax} = \mathbf{b}$. **[2 marks]**

5. Let $f : \mathbf{R} \to \mathbf{R}$ be a *quadratic* polynomial. Suppose there is an $\alpha \in \mathbf{R}$ such that

   - $f(\alpha) = 0$.
   - $f'(\alpha) \neq 0$ and $f''(\alpha) \neq 0$.

   Consider the sequence

$$x_k = x_{k-1} - \frac{f(x_{k-1}) \cdot (x_{k-1} - x_{k-2})}{f(x_{k-1}) - f(x_{k-2})}, \qquad k \geq 1,$$

   with starting values x_0, x_1.

   (a) Show that there is an $\eta > 0$ such that if x_0, $x_1 \in (\alpha - \eta, \alpha + \eta)$ then x_k is a well defined sequence of real numbers. **[2 marks]**

   (b) Let $\epsilon_k = x_k - \alpha$. Show that there are constants $M_2 \geq M_1 > 0$ such that

$$M_1|\epsilon_k||\epsilon_{k-1}| \leq |\epsilon_{k+1}| \leq M_2|\epsilon_k||\epsilon_{k-1}|,$$

   if x_0, x_1 are chosen sufficiently close to α. **[3 marks]**

   (c) For a fixed x_0, x_1 sufficiently close to α, suppose there are constants $0 < C_1 \leq C_2$ and $p > 0$ such that

$$C_1|\epsilon_k|^p \leq |\epsilon_{k+1}| \leq C_2|\epsilon_k|^p \quad \text{for all } k \geq 1.$$

   Show that $p = \frac{1+\sqrt{5}}{2}$. **[3 marks]**